APPROXIMATE METHOD FOR THE NUMERICAL SOLUTION OF UNSTEADY HEAT CONDUCTION PROBLEMS WITH VARIABLE BOUNDARY CONDITIONS OF THE THIRD KIND

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UDC 536.24

An approximate method for solving unsteady heat conduction problems with variable boundary conditions of the third order, which can be used for bodies of complex shape, is described.

The existing methods for the numerical solution of the problem of the unsteady temperature field of bodies of classical shape with boundary conditions of the third kind which vary with time [1-3] only enable the temperature field to be calculated on a computer with a large memory, with certain limitations on the assigned boundary conditions.

To investigate the temperature field of bodies of complex shape under transient conditions with boundary conditions of the third kind which vary with time, electrical simulation on grids is employed.

In view of the countless variety of possible transient conditions, it is of interest to find some approximate method of solving the problem of the unsteady temperature field for boundary conditions of the third kind which vary in an arbitrary manner, which would enable one to solve the problem for bodies of complex shape, without having to use multiple simulation, and for bodies of classical shape by the use of the simplest computer.

Consider the one-dimensional problem of unsteady heat conduction in an infinite plate

$$\frac{\partial t}{\partial \tau} = a^2 \frac{\partial^2 t}{\partial x^2} \tag{1}$$

with the following initial and boundary conditions;

$$t(0,\ x) = 0, \tag{2}$$

$$t(\tau,\ 0) = T_2, \tag{3}$$

$$\frac{\partial t(\tau,\ l)}{\partial x} = \frac{\alpha}{\lambda}(t - t_c)\Big|_{x=l}. \tag{4}$$

The solution will be sought in the form

$$t(\tau,\ x) = \int_0^\tau \frac{\mu_1(\vartheta)}{\sqrt{\pi(\tau-\vartheta)}} \exp\left\{-\frac{x^2}{4a^2(\tau-\vartheta)}\right\} d\vartheta + \int_0^\tau \frac{\mu_2(\vartheta)}{\sqrt{\pi(\tau-\vartheta)}} \exp\left\{-\frac{(x-l)^2}{4a^2(\tau-\vartheta)}\right\} d\vartheta. \tag{5}$$

By determining the functions $\mu_1(\vartheta)$ and $\mu_2(\vartheta)$ from the boundary conditions (3) and (4), and confining ourselves to the first approximation for the problem with α = const and t_c = const, we obtain

$$t(\tau,\ x) = T_2\left[1 - \Phi\left(\frac{x}{2a\sqrt{\tau}}\right)\right] - \frac{\alpha t_c(x-l)}{\lambda}\left[\frac{2a\sqrt{\tau}}{(x-l)}\exp\left\{-\frac{(x-l)^2}{4a^2\tau}\right\} + \sqrt{\pi}\left\{1 - \Phi\left(\frac{x-l}{2a\sqrt{\tau}}\right)\right\}\right]. \tag{6}$$

For $\alpha = f(\tau)$, confining ourselves to the same first approximation, we obtain

V. I. Lenin Polytechnical Institute, Kharkov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 4, pp. 688-696, September, 1969. Original article submitted September 26, 1968.

$$t(\tau,\ x)=T_2\left[1-\Phi\left(\frac{x}{2a\sqrt{\tau}}\right)\right]-\int_0^{\tau}\frac{\alpha(\vartheta)\,at_c}{\lambda\sqrt{\pi}\,(\tau-\vartheta)^{1/2}}\exp\left\{-\frac{(x-l)^2}{4a^2(\tau-\vartheta)}\right\}d\vartheta. \tag{7}$$

As can be seen, in the general case the solution has the form

$$t(\tau,\ x)=F(\tau,\ x)-\int_0^{\tau}\alpha(\vartheta)\,\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta. \tag{8}$$

If we replace the function $\alpha(\vartheta)$ under the integral by a piecewise-step function, the best approximation is obtained when

$$\int_0^{\tau}\alpha(\vartheta)\,\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta=\int_0^{\tau_1}\alpha_{a_1}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta+\int_{\tau_{1f}}^{\tau_2}\alpha_{a_2}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta+\ldots+\int_{\tau_{(n-1)f}}^{\tau_n}\alpha_{a_n}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta. \tag{9}$$

The lower limit in all the integrals, except the first, is found from the condition

$$\int_0^{\tau_{i-1}}\alpha_{a_{i-1}}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta=\int_0^{\tau_{(i-1)f}}\alpha_{a_i}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta. \tag{10}$$

Bearing in mind that

$$\int_{\tau_{(i-1)f}}^{\tau_i}\alpha_{a_i}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta=\alpha_{a_i}\int_0^{\tau_i}\varphi\left[(\tau-\vartheta),x\right]d\vartheta-\alpha_{a_i}\int_0^{\tau_{(i-1)f}}\varphi\left[(\tau-\vartheta),\ x\right]d\vartheta,$$

we obtain the approximate solution of the problem with the function $\alpha = f(\tau)$ replaced by the piecewise-step function as the algebraic sum of the solutions for α_{3i} = const, when a series of solutions τ_{if} is obtained.

We will show that, bearing in mind the uniqueness theorem, on the basis of the method for the approximate numerical solution of the problem of the unsteady temperature field for boundary conditions which vary with time for bodies of classical shape, we can use the solution obtained for constant boundary conditions.

For example, for a cylinder of infinite length, according to [4], we have

$$\bar{\theta}=\frac{t-t_{\text{н}}}{t_c-t_{\text{н}}}=1-\sum_{n=1}^{\infty}A_nJ_0\left(\mu_n\frac{r}{R}\right)\exp(-\mu_n^2\mathrm{Fo}). \tag{11}$$

Consider the problem of the heating of a cylinder of infinite length for variable boundary conditions of the third kind. We will assume that the boundary conditions on the surface of the cylinder are such that the temperature of the medium t_c remains constant and only the heat transfer coefficient $\alpha = f(\tau)$ varies with time.

Replacing $\alpha = f(\tau)$ by the piecewise-step function, as shown above, we obtain the solution in the form of the algebraic sum of solutions, if the quantities $\tau_{if}(\mathrm{Fo})$ are determined. The quantity τ_{1f} is found from the condition

$$\sum_{n=1}^{\infty}A_{n_1}J_0\left(\mu_{n_1}\frac{r}{R}\right)\exp(-\mu_{n_1}^2\mathrm{Fo}_1)=\sum_{n=1}^{\infty}A_{n_2}J_0\left(\mu_{n_2}\frac{r}{R}\right)\exp(-\mu_{n_2}^2\mathrm{Fo}_{1f}),$$

i.e., condition (10). In other words, from the value of the function $\bar{\theta}_1 = f(\mathrm{Bi}_1,\ \mathrm{Fo}_1)$ at the end of the interval in which α_{a_1} is constant, we find the fictitious time τ_{1f} for α_{a_2} to calculate $\bar{\theta} = f(\mathrm{Bi}_2,\ \mathrm{Fo})$, in the interval in which α_{a_2} is constant from τ_{1f} to $\tau_2 = \tau_{1f} + \Delta\tau_2$, etc.

Starting from this assumption, the theoretical model can be represented in the following form. By replacing the continuous function $\alpha = f(\tau)$ by a piecewise-step function in the limits of the first time interval in which α_{a_1} is constant, we determine $\bar{\theta}_1 = f(\mathrm{Bi}_1,\ \mathrm{Fo}_1)$. From the value of $\bar{\theta}_1$, reached at the end of the first time interval, we determine the fictitious time τ_{1f}, which corresponds to the same value of $\bar{\theta}_1$, but for a new value of $\mathrm{Bi}_2(\alpha_{a_2})$. Then, from the function $\bar{\theta} = f(\mathrm{Bi}_2,\ \mathrm{Fo})$ we find the value of the relative temperature in the time interval from τ_{1f} to $\tau_2 = \tau_{1f} + \Delta\tau_2$, where $\Delta\tau_2$ represents the section in which α_{a_2} is constant. From the value of $\bar{\theta}_2 = f(\mathrm{Bi}_2,\ \mathrm{Fo}_2)$ we find τ_{2f} for α_{a_3}, etc.

TABLE 1. Numerical Values of the Function $\bar{\theta}$ for Points on an Infinite Cylinder of Radius 0.1 m for a Series of Values of Bi and Fo at Radii of 0.0875 and 0.0125 m

r, m	Bi	Fo											
		0,1875	0,375	0,563	0,75	0,938	1,125	1,313	1,5	1,687	1,875	2,063	2,25
0,0875	0,3	0,135	0,232	0,310	0,382	0,445	0,502	0,552	0,597	0,638	0,677	0,709	0,739
0,0125		0,06	0,142	0,225	0,304	0,375	0,438	0,496	0,547	0,592	0,633	0,670	0,705
0,0875	1,0	0,303	0,477	0,598	0,690	0,760	0,814	0,857	0,890	0,915	0,934	0,951	0,962
0,0125		0,135	0,302	0,45	0,571	0,670	0,742	0,800	0,845	0,882	0,910	0,931	0,947
0,0875	2,0	0,440	0,641	0,760	0,839	0,891	0,926	0,950	0,965	0,977	0,985	0,990	0,993
0,0125		0,196	0,415	0,592	0,719	0,810	0,870	0,911	0,940	0,960	0,974	0,982	0,988
0,0875	3,0	0,515	0,720	0,827	0,891	0,932	0,957	0,972	0,983	0,991	0,955	0,997	0,998
0,0125		0,229	0,472	0,655	0,778	0,860	0,911	0,944	0,964	0,979	0,987	0,991	0,993
0,0875	5,0	0,601	0,797	0,888	0,937	0,963	0,979	0,989	0,994	0,996	0,997	0,998	0,999
0,0125		0,268	0,533	0,720	0,835	0,904	0,944	0,969	0,984	0,986	0,989	0,992	0,995
0,0875	10,0	0,857	0,938	0,970	0,983	0,992	0,996	0,999	0,999	0,999	0,999	0,999	0,999
0,0125		0,305	0,586	0,768	0,874	0,933	0,965	0,982	0,988	0,990	0,992	0,994	0,996

TABLE 2. Comparison of the Change in Temperature in a Cylinder of Infinite Length at Radii of 0.0875 and 0.0125 m in Heating Problems when $\alpha = f(\tau)$*

τ, min	α, W/m^2 · deg	r=0,0875 m			r=0,0125 m			α, W/m^2 · deg	r=0,0875 m			r=0,0125 m		
		$\bar{\theta}$	T_p	T_M	$\bar{\theta}$	T_p	T_M		$\bar{\theta}$	T_p	T_M	$\bar{\theta}$	T_p	T_M
2,5	116	0,13	339	340	0,06	318	318	116	0,13	339	340	0,06	318	318
5,0	116	0,23	369	370	0,14	342	343	348	0,35	405	414	0,21	363	363
7,5	348	0,43	429	433	0,297	389	386	696	0,60	480	486	0,425	427	423
10,0	348	0,565	469	473	0,45	435	429	1040	0,765	529	535	0,615	484	479
12,5	696	0,715	514	521	0,615	484	475	1740	0,875	562	568	0,76	528	524
15,0	696	0,81	543	548	0,735	522	513	3480	0,935	580	585	0,86	558	556
17,5	1040	0,88	564	570	0,832	549	542	1740	0,962	588	591	0,915	574	575
20,0	1040	0,925	577	582	0,892	567	563	1040	0,972	591	593	0,950	585	585
22,5	1740	0,96	588	591	0,935	580	578	696	0,980	594	595	0,967	590	590
25,0	1740	0,98	594	596	0,965	590	588	348	0,982	595	595	0,975	593	594
27,5	3480	0,99	597	598	0,985	595	593	116	0,983	596	596	0,982	595	595

*In the problems considered t_C = 600°K, t_H = 300°K.

TABLE 3. Comparison of the Change in Temperature in a Cylinder of Infinite Length at a Radius of 0.0875 m in Heating Problems when α = const and $t_c = f(\tau)$*

τ, min	α=348 W/m²·deg					α = 1040 W/m²·deg					α=116 W/m²·deg				
	u_{c_i}, %	$\bar{\theta}_{i_{\text{н}}}$	$\bar{\theta}_{i_{\text{к}}}$	T_p	T_M	u_{c_i}, %	$\bar{\theta}_{i_{\text{н}}}$	$\bar{\theta}_{i_{\text{к}}}$	T_p	T_M	u_{c_i}, %	$\bar{\theta}_{i_{\text{н}}}$	$\bar{\theta}_{i_{\text{к}}}$	T_p	T_M
2,5	10	0	0,303	312	312	20	0	0,515	341	341	20	0	0,13	310	310
5,0	20	0,152	0,375	330	331	25	0,412	0,68	368	368	30	0,087	0,19	323	324
7,5	30	0,250	0,440	353	356	30	0,567	0,750	390	391	40	0,143	0,238	338	340
10,0	40	0,33	0,585	394	383	35	0,642	0,785	410	414	50	0,191	0,280	356	358
12,5	50	0,468	0,60	420	414	40	0,687	0,810	430	434	60	0,233	0,316	376	379
15,0	60	0,50	0,615	447	447	45	0,720	0,830	449	455	70	0,271	0,350	398	400
17,5	70	0,525	0,635	478	480	50	0,746	0,843	469	475	80	0,307	0,382	422	424
20,0	80	0,555	0,660	511	516	55	0,767	0,855	488	496	90	0,339	0,410	448	450
22,5	90	0,586	0,680	545	552	60	0,785	0,867	508	515	100	0,369	0,435	474	477
25,0	95	0,646	0,725	576	583	65	0,800	0,875	528	536	100	0,435	0,492	496	499

*In the problems considered $t_{\text{н}}$=300°K, $t_c^{\max}$=700°K, $u_{c_i}=\dfrac{t_{c_i}-t_{\text{н}}}{t_c^{\max}-t_{\text{н}}}$, $T_p=\bar{\theta}_{i_{\text{к}}}(t_{c_i}-t_{\text{н}})+t_{\text{н}}$.

Since, when solving problems of unsteady heat conduction with variable boundary conditions on an analog computer the f functions $\alpha = f(\tau)$ are simulated by piecewise-step functions, it is of interest to compare the solutions obtained by the above approximate method with the solution obtained on an analog computer for bodies of classical shape. Moreover, since for small values of Fo the relative error in the solution of heating problems on analog computers can exceed 10-15%, to eliminate the effect of the errors introduced by the discreteness of the grid on the results obtained in comparing the numerical solution with the grid solution, in the numerical method we used as the initial function not relation (11), but the functions $\bar{\theta} = f(\text{Bi, Fo})$ for chosen points of the cylinder obtained on the analog computer in a series of problems of simple heating. The numerical values of $\bar{\theta}$ for points at radii 0.0875 and 0.0125 m on a cylinder of radius R = 0.1 m for a series of values of Bi and Fo are shown in Table 1.

The procedure described for calculating the heating conditions for a cylinder of infinite length for the case $\alpha = f(\tau)$, t_c = const was checked on examples. We compared the solutions obtained on an analog computer with the numerical solutions for a steel cylinder of radius R = 0.1 m, thermal conductivity λ = 35 W/m·deg, and a = 0.045 m²/h.

The results of the comparison are shown in Table 2. Analysis of the data shows that, to an accuracy which is satisfactory for practical calculations, the problems of heating with boundary conditions $\alpha = f(\tau)$, t_c = const can be solved numerically on a computer using relations of the type (11) for bodies of classical shape.

Since in problems of unsteady heat conduction with variable boundary conditions of the third kind a change in the temperature of the medium nearly always occurs, it is necessary to take into account the variation $t_c = f(\tau)$ in the numerical method of solution.

We will consider a number of heating problems with α = const and $t_c = f(\tau)$. In all problems the continuous function t_c is replaced by a piecewise-step function, so that in the separate intervals in which t_c is constant the change in the relative temperature of the points of the cylinder of infinite length can be found from Eq. (11). Since, at the instant when an abrupt change in the temperature of the medium occurs, the temperature of the body remains unchanged, the jump in t_c is expressed by a corresponding change in the relative temperature of the body

$$\bar{\theta}_{(i+1)_{\text{н}}} = \bar{\theta}_{i_{\text{к}}} \frac{t_{c_i} - t_{\text{н}}}{t_{c_{i+1}} - t_{\text{н}}}. \tag{12}$$

The change in the relative temperature $\bar{\theta}_i$ for α = const, due to a change in t_{c_i} indicates that the heating for a new value of $t_{c_{i+1}}$ will take place from a certain fictitious instant of time represented by the new value $\theta_{(i+1)_H}$ for α = const.

Table 3 shows the results of a comparison of the solutions obtained on an analog computer and the solutions obtained numerically for a series of values for α = const at points at radii

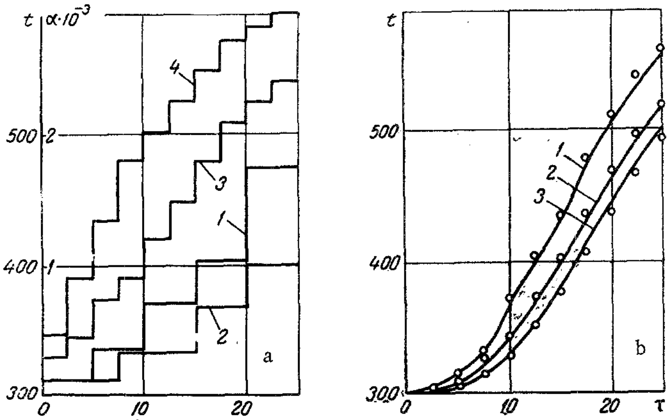

Fig. 1. Boundary conditions (a) on the surface of the cylinder, and comparison of the change in temperature (b) at points on the cylinder (Fig. 1a): a) 1) heat transfer coefficient under condition A; 2) under condition B, 3) temperature of the medium under condition A; 4) under condition B; b) 1) temperature at the radius 0.0875 m under condition B; 2) under condition A; 3) at a radius of 0.0125 m under condition A; the curves are theoretical and the points are modelling data.

0.0875 and 0.0125 m. We have considered the same cylinder as in problems of the first type. As can be seen, the value of the relative error of the numerical solution as compared with the solution obtained on the analog computer is not greater than 2%, which corresponds to the accuracy of the solution of problems of unsteady heat conduction on modelling devices.

Both the first and the second class of problems are obviously particular cases of the general problem of unsteady heat conduction with boundary conditions which vary with time, when $\alpha = f_1(\tau)$ and $t_c = f_2(\tau)$ are arbitrary. Taking into account the fact that the deviation of the assigned values of α_{ai} from the actual values introduces a smaller error into the solution than the deviation of t_{ci}, the problem can be represented in the general case as a series of problems with α_{ai} = const and $t_c = f(\tau)$. In this case the accuracy of the solution in the limits of the interval α_{ai} = const is governed by the accuracy with which t_c is approximated.

Figure 1b shows the results of a comparison of the numerical solution with the solution obtained on an analog computer for the problem of the heating of a cylinder with quite arbitrary variations of $\alpha = f_1(\tau)$ and $t_c = f_2(\tau)$ (see Fig. 1a) for points at radii of 0.0875 and 0.0125. The solutions considered for piecewise functions show that the difference between the numerical solutions using the above method and the solutions obtained on the analog computer does not exceed the normal analog computer error.

Hence, the results show that if functions of the type (11) are known at any point on the body, a numerical solution can be obtained for the variation in temperature at the given point of the body for any varying boundary conditions of the third kind. For this reason we call functions of the type (11) generalized temperature functions.

The above method of solving problems of unsteady heat conduction with varying boundary conditions of the third kind can obviously also be used for bodies of complex shape, if the generalized temperature functions $\overline{\theta} = f(\mathrm{Bi}, \mathrm{Fo})$ are known at different points. Since each function $\overline{\theta}$ represents the change in the relative temperature at the given point for constant boundary conditions α = const and t_c = const, the generalized functions for different points of a body of complex shape can be obtained by electrical simulation of a series of problems of simple heating on an analog computer.

We will consider some examples of the use of this method of calculating unsteady temperature fields in bodies of complex shape for varying boundary conditions of the third kind.

Consider the single-walled casing of a steam turbine with heat-insulating external walls and a heat supply to the internal contour only. The casing is made of 20KhMF-L steel.

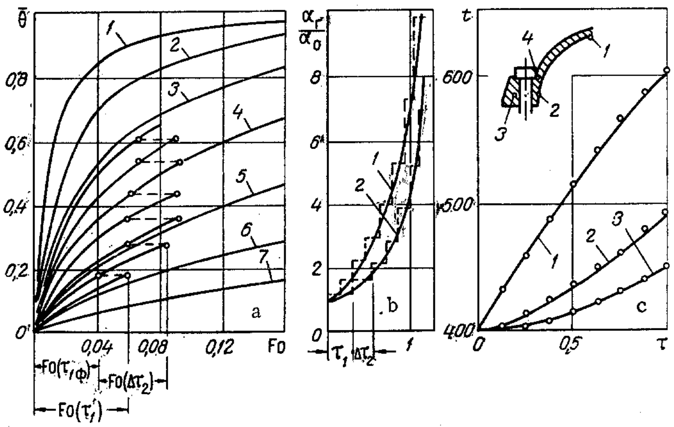

Fig. 2. Generalized functions $\bar{\theta}$ = f(Bi, Fo) at point 1 of the casing (a), the relative heat transfer coefficient α_g/α_0 at points 1 and 2 for α_0 = 29 W/m^2·deg (b), and comparison of the changes in temperature obtained by calculation and by simulation under condition A (c): a) 1) Bi = 7.92; 2) 3.96; 3) 1.98; 4) 0.94; 5) 0.47; 6) 0.234; 7) 0.177; b) 1) condition A; 2) condition B; c) 1) at point 2; 2) at point 4; 3) at point 3; the curves are theoretical and the points are simulation data.

Figure 2a shows the generalized temperature functions for point 1 on the wall of the casing, and the scheme for the numerical solution under condition 2 is shown graphically (Fig. 2b). Comparing the change in relative temperature at points 2, 3, and 4 obtained numerically, using the generalized functions $\bar{\theta}$ = f(Bi, Fo), and the method of electrical modelling of the problem with varying boundary conditions (see Fig. 2c), we see that the method of calculation satisfies practical requirements.

Consider the multistage unit-forged rotor of a steam turbine with a heat supply to the external contour, made of ÉI-415 steel. Figure 3a shows the generalized temperature functions at point 32, obtained by electrical simulation of a series of simple heating conditions. Figure 3b shows data on the heat exchange conditions in the region of this point under one of the start-up conditions, and the results of a comparison of the numerical solution using generalized functions with the solution of the same problem on an analog computer.

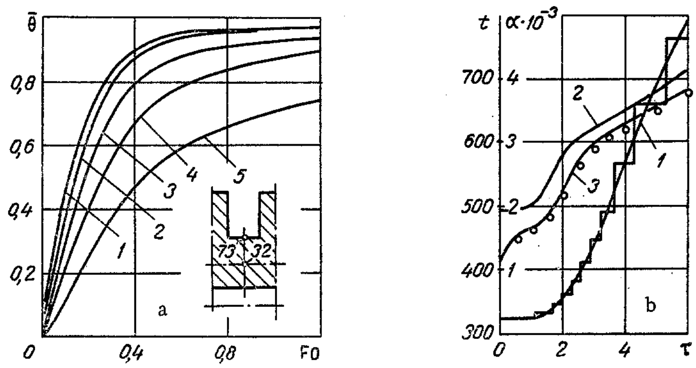

Fig. 3. Generalized functions $\bar{\theta}$ = f(Bi, Fo) at point 32 of the rotor (a), boundary conditions and change in temperature at point 32 obtained by calculation and simulation (b): a) 1) Bi = 24.5; 2) 10; 3) 5.5; 4) 2.0; 5) 0.5; b) 1) heat transfer coefficient; 2) temperature of the medium; 3) temperature of point 32; the curves are theoretical and the points are simulation data.

The examples confirm the usefulness of this method of solution since they widen the possibilities of analyzing the thermal state of the constructional elements of a motor under various transient conditions.

Moreover, the method indicates one of the ways in which the use of an analog computer and an electronic computer can be combined.

We have considered only the simplest problems involving heat transfer at a single contour for cases when $\bar{\theta} = f(\mathrm{Bi}, \mathrm{Fo})$. In order to use the method in problems of heat transfer over several contours, when the change in temperature at the given point depends not only on the heat transfer in the region of the point itself, it is necessary to develop a method of determining the generalized functions at characteristic points when heat transfer occurs at a number of contours.

NOTATION

t_{c_i}	is the temperature of the medium during the i-th interval of time;
$t_{c_{i+1}}$	is the temperature of the medium at the (i + 1)-th interval of time;
$\bar{\theta}_{iK}$	is the relative temperature at the point at the end of the i-th interval during which t_{c_i} is constant;
$\bar{\theta}_{(i+1)H}$	is the relative temperature at the point at the beginning of the (i + 1)-th interval during which $t_{c_{i+1}}$ is constant;
T_p	is the temperature of the body at the point obtained numerically;
T_M	is the temperature of the body at the point obtained by simulation;
u_{c_i}	is the potential of the medium in the i-th interval of time;
α_{a_i}	is the value of the heat transfer coefficient assigned in the i-th interval.

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